C.S.E. (MAIN)

MATHEMATICS — 2006

PAPER-I

Time allowed : 3 hours *Maximum Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any ***three*** *of the remaining questions selecting at least* **one** *question from each Section.*

Assume suitable data if considered necessary and indicate the same clearly.

The number of marks carried by each question is indicated at the end of the question.

SECTION 'A'

Q. 1. Attempt any *five* of the following :

(a) Let V be the vector space of all 2×2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V. 12

(b) State Caylay-Hamilton theorem and using it, find the inverse of

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$ 12

(c) Find a and b so that f' (2) exists, where

$$f(x) = \begin{cases} \dfrac{1}{|x|}, & \text{if } |x| > 2 \\ a + bx^2, & \text{if } |x| \le 2 \end{cases}$$ 12

(d) Express $\int_0^1 x^m (1 - x^n)^p \, dx$ in terms of Gamma function and hence evaluate the integral

$$\int_0^1 x^6 \sqrt{(1-x^2)} \, dx$$ 12

(e) A pair of tangents to the conic $ax^2 + by^2 = 1$ intercepts a constant distance 2 k on the y-axis. Prove that the locus of their point of intersection is the conic.

$$ax^2 (ax^2 + by^2 - 1) = bk^2 (ax^2 - 1)^2$$ 12

(f) Show that the length of the shortest distance between the line $z = x \tan \alpha$, $y = 0$ and any tangent to the ellipse $x^2 \sin^2 \alpha + y^2 = a^2$, $z = 0$ is constant. 12

Q. 2. (a) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$T(x, y) = (2x - 3y, x + y)$$

compute the matrix of T relative to the basis

$$\mathscr{B} = \{(1, 2), (2, 3)\}$$ 15

(b) Using elementary row operations, find the rank of the matrix

$$\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$ 15

(c) Investigate for what values of λ and μ the equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + \lambda z = \mu$$

have –

(i) no solution;

(ii) a unique solution;

(iii) infinitely many solutions. 15

(d) Find the quadratic form q (x, y) corresponding to the symmetric matrix

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 8 \end{pmatrix}$$

Is this quadratic form positive definite ? Justify your answer. 15

Q. 3. (a) Find the values of a and b such that

$$\lim_{x \to 0} \frac{a \sin^2 x \times b \log \cos x}{x^4} = \frac{1}{2}$$ 15

(b) If

$$z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$$

show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2_{xy} \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$ 15

(c) Change the order of integration in

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} \, dy \, dx$$

and hence evaluate it. 15

(d) Find the volume of the uniform ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$ 15

Q. 4. (a) If PSP′ and QSQ′ are the two perpendicular focal chords of a conic $\frac{1}{r} = 1 + e \cos \theta$, prove that

$$\frac{1}{SP \cdot SP'} + \frac{1}{SQ \cdot SQ'}$$

is constant. 15

(b) Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere

$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$$ 15

(c) Show that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines, if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$ 15

(d) If the plane $lx + my + nz = p$ passes through the extremities of three conjugate semidiameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

prove that

$$a^2 l^2 + b^2 m^2 + c^2 n^2 = 3 p^2$$ 15

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the family of curves whose tangents form an angle $\frac{\pi}{4}$ with the hyperbolas $xy = c$, $c > 0$. 12

(b) Solve the differential equation

$$\left(xy^2 + e^{-\frac{1}{x^3}}\right) dx - x^2 y \, dy = 0$$ 12

(c) A particle is free to move on a smooth vertical circular wire of radius a. It is projected horizontally from the lowest point with velocity $2\sqrt{ga}$. Show that the reaction between the particle and the wire is zero after is time

$$\sqrt{\frac{a}{g}}\log(\sqrt{5}+\sqrt{6})$$ 12

(d) The middle points of opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T′ be the tensions in these rods, prove that

$$\frac{T}{l}+\frac{T'}{l'}=0$$ 12

(e) Find the depth of the centre of pressure of a triangular lamina with a vertex in the surface of the liquid and other two vertices at depths b and c from the surface. 12

(f) Find the values of constants a, b and c so that the directional derivative of the function.

$$f = axy^2 + byz + cz^2 x^3$$

at the point (1, 2, –1) has maximum magnitude 64 in the direction parallel to z-axis. 12

Q. 6. (a) Solve :

$$(1+y^2)+(x-e^{-\tan^{-1}y})\frac{dy}{dx}=0$$ 15

(b) Solve the eqiation

$$x^2 p^2 + yp\,(2x+y)+y^2=0$$

using the substitution y = u and xy = v and find its singular solution, where

$$p=\frac{dy}{dx}$$ 15

(c) Solve the differential equation

$$x^2\frac{d^3y}{dx^3}+2x\frac{d^2y}{dx^2}+2\frac{y}{x}=10\left(1+\frac{1}{x^2}\right)$$ 15

(d) Solve the differential equation

$(D^2 - 2D + 2)\, y = e^x \tan x,\ D \equiv \dfrac{d}{dx}$ by the method of variation of parameters. 15

Q. 7. (a) A particle, whose mass is m, is acted upon by a force $m\left(x + \dfrac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance a, show that it will arrive at origin in time $\dfrac{\pi}{4}$. 15

(b) If u and V are the velocity of projection and the terminal velocity respectively of a particle rising vertically against a resistance varying as the square of the velocity, prove that the time taken by the particle to reach the highest point is

$$\frac{V}{g}\tan^{-1}\left(\frac{u}{V}\right) \qquad 15$$

(c) Show that the length of an endless chain, which will hang over a circular pulley of radius c so as to be in contact with two-third of the circumference of the pulley is

$$c\left\{\frac{3}{\log\,(2+\sqrt{3})} + \frac{4\pi}{3}\right\} \qquad 15$$

(d) A uniform rod of length 2a, can turn freely about one end, which is fixed at a height h (<2 a) above the surface of the liquid. If the densities of the rod and liquid be ρ and σ, show that the rod can rest either in a vertical position or inclined at an angle θ to the vertical such that

$$\cos\theta = \frac{h}{2a}\sqrt{\frac{\sigma}{\rho-\sigma}} \qquad 15$$

Q. 8. (a) If $\overline{A} = 2\overline{i} + \overline{k}$, $\overline{B} = \overline{i} + \overline{j} + \overline{k}$, $\overline{C} = 4\overline{i} - 3\overline{j} - 7\overline{k}$, determine a vector $\overline{R}$ satisfying the vector equations.

$$\overline{R} \times \overline{B} = \overline{C} \times \overline{B} \text{ and } \overline{R} \cdot \overline{A} = 0$$ 15

(b) Prove that $r^n \overline{r}$ is an irrotational vector for any value of n, but is solenoidal only if $n + 3 = 0$. 15

(c) If the unit tangent vector $\overline{t}$ and binormal $\overline{b}$ make angles θ and ϕ respectively with a constant unit vector $\overline{a}$, prove that

$$\frac{\sin\theta}{\sin\phi} \cdot \frac{d\theta}{d\phi} = -\frac{k}{\tau}$$ 15

(d) Verify Stokes' theorem for the function

$$\overline{F} = x^2\hat{i} - xy\hat{j}$$

integrated round the square in the plane $z = 0$ and bounded by the lines $x = 0$, $y = 0$, $x = a$ and $y = a$, $a > 0$. 15