

**Paper Specific Instructions**

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.
2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ)**. Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.
3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ)**. Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.
4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.
5. In all sections, questions not attempted will result in zero mark. In **Section – A (MCQ)**, wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B (MSQ)**, there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C (NAT)** as well.
6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.
7. The Scribble Pad will be provided for rough work.

### Special Instructions / Useful Data

|                    |   |
|--------------------|---|
| $\mathbb{R}$       | The set of all real numbers   |
| $P^T$              | Transpose of the matrix $P$   |
| $\mathbb{R}^n$     | $\left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} : x_i \in \mathbb{R}, i = 1, 2, \dots, n \right\}$  |
| $f'$               | Derivative of the differentiable function $f$   |
| $I_n$              | $n \times n$ identity matrix  |
| $P(E)$             | Probability of the event $E$  |
| $E(X)$             | Expectation of the random variable $X$  |
| $Var(X)$           | Variance of the random variable $X$   |
| i.i.d.             | Independently and identically distributed   |
| $U(a, b)$          | Continuous uniform distribution on $(a, b)$ , $-\infty < a < b < \infty$  |
| $Exp(\lambda)$     | Exponential distribution with probability density function, for $\lambda > 0$ ,<br>$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0, \\ 0, & \text{otherwise} \end{cases}$ |
| $N(\mu, \sigma^2)$ | Normal distribution with mean $\mu$ and variance $\sigma^2$   |
| $\Phi(a)$          | $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-\frac{u^2}{2}} du$  |
| $\chi_n^2$         | Central Chi-squared distribution with $n$ degrees of freedom  |
| $t_{n, \alpha}$    | A constant such that $P(X > t_{n, \alpha}) = \alpha$ , where $X$ has Student's $t$ -distribution with $n$ degrees of freedom  |
| $n!$               | $n(n-1) \cdots 3 \cdot 2 \cdot 1$ for $n = 1, 2, 3 \dots$ , and $0! = 1$  |
|                    | $\Phi(1.65) = 0.950, \quad \Phi(1.96) = 0.975$<br>$t_{4, 0.05} = 2.132, \quad t_{4, 0.10} = 1.533$  |

**SECTION – A**  
**MULTIPLE CHOICE QUESTIONS (MCQ)**

**Q. 1 – Q.10 carry one mark each.**

- Q.1 Let  $\{x_n\}_{n \geq 1}$  be a sequence of positive real numbers. Which one of the following statements is always TRUE?
- (A) If  $\{x_n\}_{n \geq 1}$  is a convergent sequence, then  $\{x_n\}_{n \geq 1}$  is monotone  
 (B) If  $\{x_n^2\}_{n \geq 1}$  is a convergent sequence, then the sequence  $\{x_n\}_{n \geq 1}$  does not converge  
 (C) If the sequence  $\{|x_{n+1} - x_n|\}_{n \geq 1}$  converges to 0, then the series  $\sum_{n=1}^{\infty} x_n$  is convergent  
 (D) If  $\{x_n\}_{n \geq 1}$  is a convergent sequence, then  $\{e^{x_n}\}_{n \geq 1}$  is also a convergent sequence
- Q.2 Consider the function  $f(x, y) = x^3 - 3xy^2$ ,  $x, y \in \mathbb{R}$ . Which one of the following statements is TRUE?
- (A)  $f$  has a local minimum at  $(0, 0)$   
 (B)  $f$  has a local maximum at  $(0, 0)$   
 (C)  $f$  has global maximum at  $(0, 0)$   
 (D)  $f$  has a saddle point at  $(0, 0)$
- Q.3 If  $F(x) = \int_{x^3}^4 \sqrt{4+t^2} dt$ , for  $x \in \mathbb{R}$ , then  $F'(1)$  equals
- (A)  $-3\sqrt{5}$                       (B)  $-2\sqrt{5}$                       (C)  $2\sqrt{5}$                       (D)  $3\sqrt{5}$
- Q.4 Let  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be a linear transformation such that  $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$  and  $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ . Suppose that  $\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix}$  and  $T\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} a \\ b \end{bmatrix}$ . Then  $\alpha + \beta + a + b$  equals
- (A)  $\frac{2}{3}$                       (B)  $\frac{4}{3}$                       (C)  $\frac{5}{3}$                       (D)  $\frac{7}{3}$
- Q.5 Two biased coins  $C_1$  and  $C_2$  have probabilities of getting heads  $\frac{2}{3}$  and  $\frac{3}{4}$ , respectively, when tossed. If both coins are tossed independently two times each, then the probability of getting exactly two heads out of these four tosses is
- (A)  $\frac{1}{4}$                       (B)  $\frac{37}{144}$                       (C)  $\frac{41}{144}$                       (D)  $\frac{49}{144}$

Q.6 Let  $X$  be a discrete random variable with the probability mass function

$$P(X = n) = \begin{cases} \frac{-2c}{n}, & n = -1, -2, \\ d, & n = 0, \\ cn, & n = 1, 2, \\ 0, & \text{otherwise,} \end{cases}$$

where  $c$  and  $d$  are positive real numbers. If  $P(|X| \leq 1) = 3/4$ , then  $E(X)$  equals

- (A)  $\frac{1}{12}$                       (B)  $\frac{1}{6}$                       (C)  $\frac{1}{3}$                       (D)  $\frac{1}{2}$

Q.7 Let  $X$  be a Poisson random variable and  $P(X = 1) + 2P(X = 0) = 12P(X = 2)$ . Which one of the following statements is TRUE?

- (A)  $0.40 < P(X = 0) \leq 0.45$                       (B)  $0.45 < P(X = 0) \leq 0.50$   
 (C)  $0.50 < P(X = 0) \leq 0.55$                       (D)  $0.55 < P(X = 0) \leq 0.60$

Q.8 Let  $X_1, X_2, \dots$  be a sequence of i.i.d. discrete random variables with the probability mass function

$$P(X_1 = m) = \begin{cases} \frac{(\log_e 2)^m}{2(m!)}, & m = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

If  $S_n = X_1 + X_2 + \dots + X_n$ , then which one of the following sequences of random variables converges to 0 in probability?

- (A)  $\frac{S_n}{n \log_e 2}$                       (B)  $\frac{S_n - n \log_e 2}{n}$                       (C)  $\frac{S_n - \log_e 2}{n}$                       (D)  $\frac{S_n - n}{\log_e 2}$

Q.9 Let  $X_1, X_2, \dots, X_n$  be a random sample from a continuous distribution with the probability density function

$$f(x) = \frac{1}{2\sqrt{2\pi}} \left[ e^{-\frac{1}{2}(x-2\mu)^2} + e^{-\frac{1}{2}(x-4\mu)^2} \right], \quad -\infty < x < \infty.$$

If  $T = X_1 + X_2 + \dots + X_n$ , then which one of the following is an unbiased estimator of  $\mu$ ?

- (A)  $\frac{T}{n}$                       (B)  $\frac{T}{2n}$                       (C)  $\frac{T}{3n}$                       (D)  $\frac{T}{4n}$

Q.10 Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $N(\theta, 1)$  distribution. Instead of observing  $X_1, X_2, \dots, X_n$ , we observe  $Y_1, Y_2, \dots, Y_n$ , where  $Y_i = e^{X_i}$ ,  $i = 1, 2, \dots, n$ . To test the hypothesis

$$H_0: \theta = 1 \text{ against } H_1: \theta \neq 1$$

based on the random sample  $Y_1, Y_2, \dots, Y_n$ , the rejection region of the likelihood ratio test is of the form, for some  $c_1 < c_2$ ,

- (A)  $\sum_{i=1}^n Y_i \leq c_1$  or  $\sum_{i=1}^n Y_i \geq c_2$       (B)  $c_1 \leq \sum_{i=1}^n Y_i \leq c_2$   
 (C)  $c_1 \leq \sum_{i=1}^n \log_e Y_i \leq c_2$       (D)  $\sum_{i=1}^n \log_e Y_i \leq c_1$  or  $\sum_{i=1}^n \log_e Y_i \geq c_2$

**Q. 11 – Q. 30 carry two marks each.**

Q.11  $\sum_{n=4}^{\infty} \frac{6}{n^2-4n+3}$  equals

- (A)  $\frac{5}{2}$       (B) 3      (C)  $\frac{7}{2}$       (D)  $\frac{9}{2}$

Q.12  $\lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{(\pi^n + e^n)^{1/n} \log_e n}$  equals

- (A)  $\frac{1}{\pi}$       (B)  $\frac{1}{e}$       (C)  $\frac{e}{\pi}$       (D)  $\frac{\pi}{e}$

Q.13 A possible value of  $b \in \mathbb{R}$  for which the equation  $x^4 + bx^3 + 1 = 0$  has no real root is

- (A)  $\frac{-11}{5}$       (B)  $\frac{-3}{2}$       (C) 2      (D)  $\frac{5}{2}$

Q.14 Let the Taylor polynomial of degree 20 for  $\frac{1}{(1-x)^3}$  at  $x = 0$  be  $\sum_{n=0}^{20} a_n x^n$ . Then  $a_{15}$  is

- (A) 136      (B) 120      (C) 60      (D) 272

Q.15 The length of the curve  $y = \frac{3}{4}x^{4/3} - \frac{3}{8}x^{2/3} + 7$  from  $x = 1$  to  $x = 8$  equals

- (A)  $\frac{99}{8}$       (B)  $\frac{117}{8}$       (C)  $\frac{99}{4}$       (D)  $\frac{117}{4}$

Q.16 The volume of the solid generated by revolving the region bounded by the parabola  $x = 2y^2 + 4$  and the line  $x = 6$  about the line  $x = 6$  is

- (A)  $\frac{78\pi}{15}$                       (B)  $\frac{91\pi}{15}$                       (C)  $\frac{64\pi}{15}$                       (D)  $\frac{117\pi}{15}$

Q.17 Let  $P$  be a  $3 \times 3$  non-null real matrix. If there exist a  $3 \times 2$  real matrix  $Q$  and a  $2 \times 3$  real matrix  $R$  such that  $P = QR$ , then

- (A)  $P\mathbf{x} = \mathbf{0}$  has a unique solution, where  $\mathbf{0} \in \mathbb{R}^3$   
 (B) there exists  $\mathbf{b} \in \mathbb{R}^3$  such that  $P\mathbf{x} = \mathbf{b}$  has no solution  
 (C) there exists a non-zero  $\mathbf{b} \in \mathbb{R}^3$  such that  $P\mathbf{x} = \mathbf{b}$  has a unique solution  
 (D) there exists a non-zero  $\mathbf{b} \in \mathbb{R}^3$  such that  $P^T\mathbf{x} = \mathbf{b}$  has a unique solution

Q.18 If  $P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$  and  $6P^{-1} = aI_3 + bP - P^2$ , then the ordered pair  $(a, b)$  is

- (A) (3, 2)                      (B) (2, 3)                      (C) (4, 5)                      (D) (5, 4)

Q.19 Let  $E, F$  and  $G$  be any three events with  $P(E) = 0.3$ ,  $P(F|E) = 0.2$ ,  $P(G|E) = 0.1$  and  $P(F \cap G|E) = 0.05$ . Then  $P(E - (F \cup G))$  equals

- (A) 0.155                      (B) 0.175                      (C) 0.225                      (D) 0.255

Q.20 Let  $E$  and  $F$  be any two independent events with  $0 < P(E) < 1$  and  $0 < P(F) < 1$ . Which one of the following statements is **NOT** TRUE?

- (A)  $P(\text{Neither } E \text{ nor } F \text{ occurs}) = (P(E) - 1)(P(F) - 1)$   
 (B)  $P(\text{Exactly one of } E \text{ and } F \text{ occurs}) = P(E) + P(F) - P(E)P(F)$   
 (C)  $P(E \text{ occurs but } F \text{ does not occur}) = P(E) - P(E \cap F)$   
 (D)  $P(E \text{ occurs given that } F \text{ does not occur}) = P(E)$

Q.21 Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \begin{cases} \frac{1}{3} x^7 e^{-x^2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then the distribution of the random variable  $W = 2X^2$  is

- (A)  $\chi_2^2$                       (B)  $\chi_4^2$                       (C)  $\chi_6^2$                       (D)  $\chi_8^2$

Q.22 Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \frac{e^x}{(1 + e^x)^2}, \quad -\infty < x < \infty.$$

Then  $E(X)$  and  $P(X > 1)$ , respectively, are

- (A) 1 and  $(1 + e)^{-1}$                       (B) 0 and  $2(1 + e)^{-2}$   
(C) 2 and  $(2 + 2e)^{-1}$                       (D) 0 and  $(1 + e)^{-1}$

Q.23 The lifetime (in years) of bulbs is distributed as an  $Exp(1)$  random variable. Using Poisson approximation to the binomial distribution, the probability (round off to 2 decimal places) that out of the fifty randomly chosen bulbs at most one fails within one month equals

- (A) 0.05                      (B) 0.07                      (C) 0.09                      (D) 0.11

Q.24 Let  $X$  follow a beta distribution with parameters  $m (> 0)$  and 2. If  $P\left(X \leq \frac{1}{2}\right) = \frac{1}{2}$ , then  $Var(X)$  equals

- (A)  $\frac{1}{10}$                       (B)  $\frac{1}{20}$                       (C)  $\frac{1}{25}$                       (D)  $\frac{1}{40}$

Q.25 Let  $X_1, X_2$  and  $X_3$  be i.i.d.  $U(0,1)$  random variables. Then  $P(X_1 > X_2 + X_3)$  equals

- (A)  $\frac{1}{6}$                       (B)  $\frac{1}{4}$                       (C)  $\frac{1}{3}$                       (D)  $\frac{1}{2}$

Q.26 Let  $X$  and  $Y$  be i.i.d.  $U(0, 1)$  random variables. Then  $E(X|X > Y)$  equals

- (A)  $\frac{1}{3}$                       (B)  $\frac{1}{2}$                       (C)  $\frac{2}{3}$                       (D)  $\frac{3}{4}$

Q.27 Let  $-1$  and  $1$  be the observed values of a random sample of size two from  $N(\theta, \theta)$  distribution. The maximum likelihood estimate of  $\theta$  is

- (A) 0                      (B) 2                      (C)  $\frac{-\sqrt{5}-1}{2}$                       (D)  $\frac{\sqrt{5}-1}{2}$

- Q.28 Let  $X_1$  and  $X_2$  be a random sample from a continuous distribution with the probability density function

$$f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x-\theta}{\theta}}, & x > \theta, \\ 0, & \text{otherwise,} \end{cases}$$

where  $\theta > 0$ . If  $X_{(1)} = \min\{X_1, X_2\}$  and  $\bar{X} = \frac{(X_1 + X_2)}{2}$ , then which one of the following statements is TRUE?

- (A)  $(\bar{X}, X_{(1)})$  is sufficient and complete  
(B)  $(\bar{X}, X_{(1)})$  is sufficient but not complete  
(C)  $(\bar{X}, X_{(1)})$  is complete but not sufficient  
(D)  $(\bar{X}, X_{(1)})$  is neither sufficient nor complete
- Q.29 Let  $X_1, X_2, \dots, X_n$  be a random sample from a continuous distribution with the probability density function  $f(x)$ . To test the hypothesis  
 $H_0: f(x) = \frac{1}{\sqrt{\pi}} e^{-x^2}, -\infty < x < \infty$  against  $H_1: f(x) = e^{-2|x|}, -\infty < x < \infty$ ,  
the rejection region of the most powerful size  $\alpha$  test is of the form, for some  $c > 0$ ,
- (A)  $\sum_{i=1}^n (X_i - 1)^2 \geq c$  (B)  $\sum_{i=1}^n (X_i - 1)^2 \leq c$   
(C)  $\sum_{i=1}^n (|X_i| - 1)^2 \geq c$  (D)  $\sum_{i=1}^n (|X_i| - 1)^2 \leq c$

- Q.30 Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $N(\theta, 1)$  distribution. To test  $H_0: \theta = 0$  against  $H_1: \theta = 1$ , assume that the critical region is given by  $\frac{1}{n} \sum_{i=1}^n X_i > \frac{3}{4}$ . Then the minimum sample size required so that  $P(\text{Type I error}) \leq 0.05$  is
- (A) 3 (B) 4 (C) 5 (D) 6



## SECTION - B

## MULTIPLE SELECT QUESTIONS (MSQ)

**Q. 31 – Q. 40 carry two marks each.**

- Q.31 Let  $\{x_n\}_{n \geq 1}$  be a sequence of positive real numbers such that the series  $\sum_{n=1}^{\infty} x_n$  converges. Which of the following statements is (are) always TRUE?
- (A) The series  $\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$  converges
- (B)  $\lim_{n \rightarrow \infty} n x_n = 0$
- (C) The series  $\sum_{n=1}^{\infty} \sin^2 x_n$  converges
- (D) The series  $\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{1+\sqrt{x_n}}$  converges
- Q.32 Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be continuous on  $\mathbb{R}$  and differentiable on  $(-\infty, 0) \cup (0, \infty)$ . Which of the following statements is (are) always TRUE?
- (A) If  $f$  is differentiable at 0 and  $f'(0) = 0$ , then  $f$  has a local maximum or a local minimum at 0
- (B) If  $f$  has a local minimum at 0, then  $f$  is differentiable at 0 and  $f'(0) = 0$
- (C) If  $f'(x) < 0$  for all  $x < 0$  and  $f'(x) > 0$  for all  $x > 0$ , then  $f$  has a global maximum at 0
- (D) If  $f'(x) > 0$  for all  $x < 0$  and  $f'(x) < 0$  for all  $x > 0$ , then  $f$  has a global maximum at 0
- Q.33 Let  $P$  be a  $2 \times 2$  real matrix such that every non-zero vector in  $\mathbb{R}^2$  is an eigenvector of  $P$ . Suppose that  $\lambda_1$  and  $\lambda_2$  denote the eigenvalues of  $P$  and  $P \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 2 \\ t \end{bmatrix}$  for some  $t \in \mathbb{R}$ . Which of the following statements is (are) TRUE?
- (A)  $\lambda_1 \neq \lambda_2$
- (B)  $\lambda_1 \lambda_2 = 2$
- (C)  $\sqrt{2}$  is an eigenvalue of  $P$
- (D)  $\sqrt{3}$  is an eigenvalue of  $P$
- Q.34 Let  $P$  be an  $n \times n$  non-null real skew-symmetric matrix, where  $n$  is even. Which of the following statements is (are) always TRUE?
- (A)  $P\mathbf{x} = \mathbf{0}$  has infinitely many solutions, where  $\mathbf{0} \in \mathbb{R}^n$
- (B)  $P\mathbf{x} = \lambda\mathbf{x}$  has a unique solution for every non-zero  $\lambda \in \mathbb{R}$
- (C) If  $Q = (I_n + P)(I_n - P)^{-1}$ , then  $Q^T Q = I_n$
- (D) The sum of all the eigenvalues of  $P$  is zero

Q.35 Let  $X$  be a random variable with the cumulative distribution function

$$F(x) = \begin{cases} 0, & x < 0, \\ \frac{1+x^2}{10}, & 0 \leq x < 1, \\ \frac{3+x^2}{10}, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

Which of the following statements is (are) TRUE?

- (A)  $P(1 < X < 2) = \frac{3}{10}$                       (B)  $P(1 < X \leq 2) = \frac{3}{5}$   
 (C)  $P(1 \leq X < 2) = \frac{1}{2}$                       (D)  $P(1 \leq X \leq 2) = \frac{4}{5}$

Q.36 Let  $X$  and  $Y$  be i.i.d.  $Exp(\lambda)$  random variables. If  $Z = \max\{X - Y, 0\}$ , then which of the following statements is (are) TRUE?

- (A)  $P(Z = 0) = \frac{1}{2}$   
 (B) The cumulative distribution function of  $Z$  is  $F(z) = \begin{cases} 0, & z < 0, \\ 1 - \frac{1}{2}e^{-\lambda z}, & z \geq 0 \end{cases}$   
 (C)  $P(Z = 0) = 0$   
 (D) The cumulative distribution function of  $Z$  is  $F(z) = \begin{cases} 0, & z < 0, \\ 1 - e^{-\lambda z/2}, & z \geq 0 \end{cases}$

Q.37 Let the discrete random variables  $X$  and  $Y$  have the joint probability mass function

$$P(X = m, Y = n) = \begin{cases} \frac{e^{-2}}{m! n!}, & m = 0, 1, 2, \dots; n = 0, 1, 2, \dots, \\ 0, & \text{otherwise.} \end{cases}$$

Which of the following statements is (are) TRUE?

- (A) The marginal distribution of  $X$  is Poisson with mean 2  
 (B) The random variables  $X$  and  $Y$  are independent  
 (C) The covariance between  $X$  and  $X + \sqrt{3}Y$  is 1  
 (D)  $P(Y = n) = (n + 1)P(Y = n + 1)$  for  $n = 0, 1, 2, \dots$

- Q.38 Let  $X_1, X_2, \dots$  be a sequence of i.i.d. continuous random variables with the probability density function

$$f(x) = \begin{cases} 2e^{-2(x-\frac{1}{2})}, & x \geq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

If  $S_n = X_1 + X_2 + \dots + X_n$  and  $\bar{X}_n = S_n/n$ , then the distributions of which of the following sequences of random variables converge(s) to a normal distribution with mean 0 and a finite variance?

- (A)  $\frac{S_n - n}{\sqrt{n}}$       (B)  $\frac{S_n}{\sqrt{n}}$       (C)  $\sqrt{n}(\bar{X}_n - 1)$       (D)  $\frac{\sqrt{n}(\bar{X}_n - 1)}{2}$

- Q.39 Let  $X_1, X_2, \dots, X_n$  be a random sample from a  $U(\theta, 0)$  distribution, where  $\theta < 0$ . If  $T_n = \min\{X_1, X_2, \dots, X_n\}$ , then which of the following sequences of estimators is (are) consistent for  $\theta$ ?

- (A)  $T_n$       (B)  $T_n - 1$       (C)  $T_n + \frac{1}{n}$       (D)  $T_n - 1 - \frac{1}{n^2}$

- Q.40 Let  $X_1, X_2, \dots, X_n$  be a random sample from a continuous distribution with the probability density function, for  $\lambda > 0$ ,

$$f(x) = \begin{cases} 2\lambda x e^{-\lambda x^2}, & x > 0, \\ 0, & \text{otherwise.} \end{cases}$$

To test the hypothesis  $H_0: \lambda = \frac{1}{2}$  against  $H_1: \lambda = \frac{3}{4}$  at the level  $\alpha$  ( $0 < \alpha < 1$ ), which of the following statements is (are) TRUE?

- (A) The most powerful test exists for each value of  $\alpha$   
 (B) The most powerful test does not exist for some values of  $\alpha$   
 (C) If the most powerful test exists, it is of the form: Reject  $H_0$  if  $X_1^2 + X_2^2 + \dots + X_n^2 \leq c$  for some  $c > 0$   
 (D) If the most powerful test exists, it is of the form: Reject  $H_0$  if  $X_1^2 + X_2^2 + \dots + X_n^2 \geq c$  for some  $c > 0$

## SECTION – C

## NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41  $\lim_{n \rightarrow \infty} \frac{\sqrt{n+1} + \sqrt{n+2} + \dots + \sqrt{n+n}}{n\sqrt{n}}$  (round off to 2 decimal places) equals \_\_\_\_\_

Q.42 Let  $f: [0, 2] \rightarrow \mathbb{R}$  be such that  $|f(x) - f(y)| \leq |x - y|^{4/3}$  for all  $x, y \in [0, 2]$ .  
If  $\int_0^2 f(x) dx = \frac{2}{3}$ , then  $\sum_{k=1}^{2019} f\left(\frac{1}{k}\right)$  equals \_\_\_\_\_

Q.43 The value (round off to 2 decimal places) of the double integral

$$\int_0^9 \int_{\sqrt{x}}^3 \frac{1}{1+y^3} dy dx$$

equals \_\_\_\_\_

Q.44 If  $\begin{bmatrix} \frac{\sqrt{5}}{3} & -\frac{2}{3} & c \\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d \\ a & b & 1 \end{bmatrix}$  is a real orthogonal matrix, then  $a^2 + b^2 + c^2 + d^2$  equals \_\_\_\_\_

Q.45 Two fair dice are tossed independently and it is found that one face is odd and the other one is even. Then the probability (round off to 2 decimal places) that the sum is less than 6 equals \_\_\_\_\_

Q.46 Let  $X$  be a random variable with the moment generating function

$$M_X(t) = \left( \frac{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}{2} \right)^2, \quad -\infty < t < \infty.$$

Using Chebyshev's inequality, the upper bound for  $P\left(|X| > \sqrt{\frac{2}{3}}\right)$  equals \_\_\_\_\_

- Q.47 In a production line of a factory, each packet contains four items. Past record shows that 20% of the produced items are defective. A quality manager inspects each item in a packet and approves the packet for shipment if at most one item in the packet is found to be defective. Then the probability (round off to 2 decimal places) that out of the three randomly inspected packets at least two are approved for shipment equals \_\_\_\_\_
- Q.48 Let  $X$  be the number of heads obtained in a sequence of 10 independent tosses of a fair coin. The fair coin is tossed again  $X$  number of times independently, and let  $Y$  be the number of heads obtained in these  $X$  number of tosses. Then  $E(X + 2Y)$  equals \_\_\_\_\_
- Q.49 Let  $0, 1, 0, 0, 1$  be the observed values of a random sample of size five from a discrete distribution with the probability mass function  $P(X = 1) = 1 - P(X = 0) = 1 - e^{-\lambda}$ , where  $\lambda > 0$ . The method of moments estimate (round off to 2 decimal places) of  $\lambda$  equals \_\_\_\_\_
- Q.50 Let  $X_1, X_2, X_3$  be a random sample from  $N(\mu_1, \sigma^2)$  distribution and  $Y_1, Y_2, Y_3$  be a random sample from  $N(\mu_2, \sigma^2)$  distribution. Also, assume that  $(X_1, X_2, X_3)$  and  $(Y_1, Y_2, Y_3)$  are independent. Let the observed values of  $\sum_{i=1}^3 \left[ X_i - \frac{1}{3}(X_1 + X_2 + X_3) \right]^2$  and  $\sum_{i=1}^3 \left[ Y_i - \frac{1}{3}(Y_1 + Y_2 + Y_3) \right]^2$  be 1 and 5, respectively. The length (round off to 2 decimal places) of the shortest 90% confidence interval of  $\mu_1 - \mu_2$  equals \_\_\_\_\_

**Q. 51 – Q. 60 carry two marks each.**

- Q.51  $\lim_{n \rightarrow \infty} \left[ n - \frac{n}{e} \left( 1 + \frac{1}{n} \right)^n \right]$  equals \_\_\_\_\_
- Q.52 For any real number  $y$ , let  $[y]$  be the greatest integer less than or equal to  $y$  and let  $\{y\} = y - [y]$ . For  $n = 1, 2, \dots$ , and for  $x \in \mathbb{R}$ , let
- $$f_{2n}(x) = \begin{cases} \left[ \frac{\sin x}{x} \right], & x \neq 0, \\ 1, & x = 0, \end{cases} \quad \text{and} \quad f_{2n-1}(x) = \begin{cases} \left\{ \frac{\sin x}{x} \right\}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$
- Then  $\lim_{x \rightarrow 0} \sum_{k=1}^{100} f_k(x)$  equals \_\_\_\_\_

Q.53 The volume (round off to 2 decimal places) of the region in the first octant ( $x \geq 0, y \geq 0, z \geq 0$ ) bounded by the cylinder  $x^2 + y^2 = 4$  and the planes  $z = 2$  and  $y + z = 4$  equals \_\_\_\_\_

Q.54

If  $ad - bc = 2$  and  $ps - qr = 1$ , then the determinant of  $\begin{bmatrix} a & b & 0 & 0 \\ 3 & 10 & 2p & q \\ c & d & 0 & 0 \\ 2 & 7 & 2r & s \end{bmatrix}$  equals \_\_\_\_\_

Q.55 In an ethnic group, 30% of the adult male population is known to have heart disease. A test indicates high cholesterol level in 80% of adult males with heart disease. But the test also indicates high cholesterol levels in 10% of the adult males with no heart disease. Then the probability (round off to 2 decimal places), that a randomly selected adult male from this population does not have heart disease given that the test indicates high cholesterol level, equals \_\_\_\_\_

Q.56 Let  $X$  be a continuous random variable with the probability density function

$$f(x) = \begin{cases} ax^2, & 0 < x < 1, \\ bx^{-4}, & x \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

where  $a$  and  $b$  are positive real numbers. If  $E(X) = 1$ , then  $E(X^2)$  equals \_\_\_\_\_

Q.57 Let  $X$  and  $Y$  be jointly distributed continuous random variables, where  $Y$  is positive valued with  $E(Y^2) = 6$ . If the conditional distribution of  $X$  given  $Y = y$  is  $U(1 - y, 1 + y)$ , then  $Var(X)$  equals \_\_\_\_\_

Q.58 Let  $X_1, X_2, \dots, X_{10}$  be i.i.d.  $N(0, 1)$  random variables. If  $T = X_1^2 + X_2^2 + \dots + X_{10}^2$ , then  $E\left(\frac{1}{T}\right)$  equals \_\_\_\_\_

Q.59 Let  $X_1, X_2, X_3$  be a random sample from a continuous distribution with the probability density function

$$f(x) = \begin{cases} e^{-(x-\mu)}, & x > \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Let  $X_{(1)} = \min\{X_1, X_2, X_3\}$  and  $c > 0$  be a real number. Then  $(X_{(1)} - c, X_{(1)})$  is a 97% confidence interval for  $\mu$ , if  $c$  (round off to 2-decimal places) equals \_\_\_\_\_

Q.60 Let  $X_1, X_2, X_3, X_4$  be a random sample from a discrete distribution with the probability mass function  $P(X = 0) = 1 - P(X = 1) = 1 - p$ , for  $0 < p < 1$ . To test the hypothesis

$$H_0: p = \frac{3}{4} \quad \text{against} \quad H_1: p = \frac{4}{5},$$

consider the test:

Reject  $H_0$  if  $X_1 + X_2 + X_3 + X_4 > 3$ .

Let the size and power of the test be denoted by  $\alpha$  and  $\gamma$ , respectively. Then  $\alpha + \gamma$  (round off to 2 decimal places) equals \_\_\_\_\_

**END OF THE QUESTION PAPER**