Paper Specific Instructions

1. The examination is of 3 hours duration. There are a total of 60 questions carrying 100 marks. The entire paper is divided into three sections, **A**, **B** and **C**. All sections are compulsory. Questions in each section are of different types.

2. **Section – A** contains a total of 30 **Multiple Choice Questions (MCQ).** Each MCQ type question has four choices out of which only **one** choice is the correct answer. Questions Q.1 – Q.30 belong to this section and carry a total of 50 marks. Q.1 – Q.10 carry 1 mark each and Questions Q.11 – Q.30 carry 2 marks each.

3. **Section – B** contains a total of 10 **Multiple Select Questions (MSQ).** Each MSQ type question is similar to MCQ but with a difference that there may be **one or more than one** choice(s) that are correct out of the four given choices. The candidate gets full credit if he/she selects all the correct answers only and no wrong answers. Questions Q.31 – Q.40 belong to this section and carry 2 marks each with a total of 20 marks.

4. **Section – C** contains a total of 20 **Numerical Answer Type (NAT)** questions. For these NAT type questions, the answer is a real number which needs to be entered using the virtual keyboard on the monitor. No choices will be shown for these type of questions. Questions Q.41 – Q.60 belong to this section and carry a total of 30 marks. Q.41 – Q.50 carry 1 mark each and Questions Q.51 – Q.60 carry 2 marks each.

5. In all sections, questions not attempted will result in zero mark. In **Section – A** (MCQ), wrong answer will result in **NEGATIVE** marks. For all 1 mark questions, 1/3 marks will be deducted for each wrong answer. For all 2 marks questions, 2/3 marks will be deducted for each wrong answer. In **Section – B** (MSQ), there is **NO NEGATIVE** and **NO PARTIAL** marking provisions. There is **NO NEGATIVE** marking in **Section – C** (NAT) as well.

6. Only Virtual Scientific Calculator is allowed. Charts, graph sheets, tables, cellular phone or other electronic gadgets are **NOT** allowed in the examination hall.

7. The Scribble Pad will be provided for rough work.

Special Instructions / Useful Data

$\mathbb{R}$	The set of all real numbers
$\mathbb{R}^n$	$\{(x_1, x_2, \dots, x_n): x_i \in \mathbb{R}, i = 1,2, \dots, n\}, n = 2, 3, \dots$
f'	First order derivative of the differentiable function f
M^{-1}	Inverse of the non-singular matrix M
$\det(M)$	Determinant of the square matrix M
$\text{trace}(M)$	Trace of the square matrix M
I_n	Identity matrix of order $n \times n$, $n = 2, 3, \dots$
$P(H)$	Probability of event H
H^c	Complement of an event H
$E(X)$	Expectation of the random variable X
$\text{Var}(X)$	Variance of the random variable X
i.i.d.	Independently and identically distributed
$\text{Bin}(n, p)$	Binomial distribution with parameters n and p, $n = 1, 2, \dots$ and $p \in (0, 1)$
$\text{Poisson}(\lambda)$	Poisson distribution with mean λ, $\lambda > 0$
$U(a, b)$	Continuous uniform distribution on the interval $(a, b), -\infty < a < b < \infty$
$\text{Exp}(\lambda)$	Exponential distribution with probability density function $f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}, \quad \lambda > 0$
$N(\mu, \sigma^2)$	Normal distribution with mean μ and variance σ^2, $\mu \in \mathbb{R}$ and $\sigma > 0$
χ_n^2	Central Chi-squared distribution with n degrees of freedom, $n = 1, 2, \dots$
t_n	Central t-distribution with n degrees of freedom, $n = 1, 2, \dots$
$\chi_{n,\alpha}^2$	For $\alpha \in (0, 1)$ and positive integer n, $P\left(Y_n > \chi_{n,\alpha}^2\right) = \alpha$, where $Y_n \sim \chi_n^2$
$F_{m,n}$	Central F distribution with m and n degrees of freedom; $m, n = 1, 2, \dots$
$f_{m,n,\alpha}$	For $\alpha \in (0, 1)$ and positive integers m and n, $P\left(F_{m,n} > f_{m,n,\alpha}\right) = \alpha$
$n!$	$n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1, \quad n = 1, 2, \dots$ and $0! = 1$
$\binom{n}{k}$	$\frac{n!}{(n-k)!\,k!}, k = 0,1, \dots, n$ and $n = 1, 2, \dots$

$$\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} \, dt, -\infty < a < \infty$$

$$\Phi(0.5) = 0.6915, \Phi(1) = 0.8413, \Phi(1.5) = 0.9332, \Phi(2) = 0.9772$$

SECTION – A

MULTIPLE CHOICE QUESTIONS (MCQ)

Q. 1 – Q.10 carry one mark each.

Q.1 If $\{x_n\}_{n\geq 1}$is a sequence of real numbers such that $\lim_{n\to\infty} \frac{x_n}{n} = 0.001$, then

(A) $\{x_n\}_{n\geq 1}$ is a bounded sequence

(B) $\{x_n\}_{n\geq 1}$ is an unbounded sequence

(C) $\{x_n\}_{n\geq 1}$ is a convergent sequence

(D) $\{x_n\}_{n\geq 1}$ is a monotonically decreasing sequence

Q.2 For real constants a and b, let

$$f(x) = \begin{cases} \dfrac{a \sin x - 2x}{x}, & x < 0 \\ bx, & x \geq 0 \end{cases}$$

If f is a differentiable function then the value of $a + b$ is

(A) 0 (B) 1 (C) 2 (D) 3

Q.3 The area of the region bounded by the curves $y_1(x) = x^4 - 2x^2$ and $y_2(x) = 2x^2$, $x \in \mathbb{R}$, is

(A) $\frac{128}{15}$ (B) $\frac{129}{15}$ (C) $\frac{133}{15}$ (D) $\frac{134}{15}$

Q.4 Consider the following system of linear equations

$$\begin{aligned} ax + 2y + z &= 0 \\ y + 5z &= 1 \\ by - 5z &= -1 \end{aligned}$$

Which one of the following statements is TRUE?

(A) The system has unique solution for $a = 1, b = -1$

(B) The system has unique solution for $a = -1, b = 1$

(C) The system has no solution for $a = 1, b = 0$

(D) The system has infinitely many solutions for $a = 0, b = 0$

Q.5 Let E and F be two events. Then which one of the following statements is **NOT** always TRUE?

(A) $P(E \cap F) \leq \max\{1 - P(E^c) - P(F^c), 0\}$ (B) $P(E \cup F) \geq \max\{P(E), P(F)\}$

(C) $P(E \cup F) \leq \min\{ P(E) + P(F), 1\}$ (D) $P(E \cap F) \leq \min\{P(E), P(F)\}$

Q.6 Let X be a random variable having $\text{Poisson}(2)$ distribution. Then $E\left(\frac{1}{1+X}\right)$ equals

(A) $1-e^{-2}$ (B) e^{-2} (C) $\frac{1}{2}(1-e^{-2})$ (D) $\frac{1}{2}e^{-1}$

Q.7 The mean and the standard deviation of weights of ponies in a large animal shelter are 20 kg and 3 kg, respectively. A pony is selected at random from the shelter. Using Chebyshev's inequality, the value of the lower bound of the probability that the weight of the selected pony is between 14 kg and 26 kg is

(A) $\frac{3}{4}$ (B) $\frac{1}{4}$ (C) 0 (D) 1

Q.8 Let $X_1, X_2, \ldots, X_{10}$ be a random sample from $N(1,2)$ distribution. If

$$\bar{X} = \frac{1}{10}\sum_{i=1}^{10} X_i \quad \text{and} \quad S^2 = \frac{1}{9}\sum_{i=1}^{10}(X_i - \bar{X})^2,$$

then $\text{Var}(S^2)$ equals

(A) $\frac{2}{5}$ (B) $\frac{4}{9}$ (C) $\frac{11}{9}$ (D) $\frac{8}{9}$

Q.9 Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables such that $E(X_i) = 1$ and $\text{Var}(X_i) = 1$, $i = 1, 2, \ldots$. Then the approximate distribution of $\frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_{2i} - X_{2i-1})$, for large n, is

(A) $N(0,1)$ (B) $N(0,2)$ (C) $N(0,0.5)$ (D) $N(0,0.25)$

Q.10 Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables having $N(\mu, \sigma^2)$ distribution, where $\mu \in \mathbb{R}$ and $\sigma > 0$. Define

$$W = \frac{1}{2n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(X_i - X_j\right)^2$$

Then W, as an estimator of σ^2, is

(A) biased and consistent (B) unbiased and consistent

(C) biased and inconsistent (D) unbiased and inconsistent

Q. 11 – Q. 30 carry two marks each.

Q.11 Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 1, a_2 = 7$ and $a_{n+1} = \frac{a_n + a_{n-1}}{2}, n \geq 2$. Assuming that $\lim\limits_{n\to\infty} a_n$ exists, the value of $\lim\limits_{n\to\infty} a_n$ is

(A) $\frac{19}{4}$ (B) $\frac{9}{2}$ (C) 5 (D) $\frac{21}{4}$

Q.12 Which one of the following series is convergent?

(A) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{5n+1}{4n+1}\right)^n$

(B) $\displaystyle\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

(C) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin\left(\frac{1}{n}\right)}{n^{1/n}}$

(D) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n}\left(1 - \cos\left(\frac{1}{n}\right)\right)$

Q.13 Let α and β be two real numbers. If

$$\lim_{x\to 0} \frac{\tan 2x - 2\sin \alpha x}{x(1 - \cos 2x)} = \beta$$

then $\alpha + \beta$ equals

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

Q.14 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{2x^3 + 3y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Let $f_x(0,0)$ and $f_y(0,0)$ denote the first order partial derivatives of $f(x, y)$ with respect to x and y, respectively, at the point $(0,0)$. Then which one of the following statements is TRUE?

(A) f is continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ do not exist

(B) f is differentiable at $(0,0)$

(C) f is not differentiable at $(0,0)$

(D) f is not continuous at $(0,0)$ but $f_x(0,0)$ and $f_y(0,0)$ exist

Q.15 If the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 2y$ is given by

$$\int_0^{\alpha} \int_{\beta(y)}^{2y} \int_{-\sqrt{z-y^2}}^{\sqrt{z-y^2}} dx\, dz\, dy$$

then

(A) $\alpha = 2$ and $\beta(y) = y,\ y \in [0,2]$ (B) $\alpha = 1$ and $\beta(y) = y^2,\ y \in [0,1]$

(C) $\alpha = 2$ and $\beta(y) = y^2,\ y \in [0,2]$ (D) $\alpha = 1$ and $\beta(y) = y,\ y \in [0,1]$

Q.16 The value of the integral

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\ dy\, dx$$

is

(A) $\frac{17}{9}$ (B) $\frac{16}{9}$ (C) $\frac{14}{9}$ (D) $\frac{13}{9}$

Q.17 Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be a linear transformation. If $T(1,1,0) = (2,0,0,0)$, $T(1,0,1) = (2,4,0,0)$ and $T(0,1,1) = (0,0,2,0)$, then $T(1,1,1)$ equals

(A) $(1,1,1,0)$ (B) $(0,1,1,1)$ (C) $(2,2,1,0)$ (D) $(0,0,0,0)$

Q.18 Let M be an $n \times n$ non-zero skew symmetric matrix. Then the matrix $(I_n - M)(I_n + M)^{-1}$ is always

(A) singular (B) symmetric (C) orthogonal (D) idempotent

Q.19 A packet contains 10 distinguishable firecrackers out of which 4 are defective. If three firecrackers are drawn at random (without replacement) from the packet, then the probability that all three firecrackers are defective equals

(A) $\frac{1}{10}$ (B) $\frac{1}{20}$ (C) $\frac{1}{30}$ (D) $\frac{1}{40}$

Q.20 Let X_1, X_2, X_3, X_4 be i.i.d. random variables having a continuous distribution. Then $P(X_3 < X_2 < \max(X_1, X_4))$ equals

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{6}$

Q.21 Consider the simple linear regression model

$$y_i = \alpha + \beta x_i + \epsilon_i, \qquad i = 1, 2, \ldots, n,$$

where $\epsilon_i{}'s$ are i.i.d. random variables with mean 0 and variance $\sigma^2 \in (0, \infty)$. Suppose that we have a data set $(x_1, y_1), \ldots, (x_n, y_n)$ with $n = 20$, $\sum_{i=1}^{n} x_i = 100$, $\sum_{i=1}^{n} y_i = 50$, $\sum_{i=1}^{n} x_i^2 = 600$, $\sum_{i=1}^{n} y_i^2 = 500$ and $\sum_{i=1}^{n} x_i y_i = 400$. Then the least square estimates of α and β are, respectively,

(A) 5 and $\frac{3}{2}$ (B) -5 and $\frac{3}{2}$ (C) 5 and $-\frac{3}{2}$ (D) -5 and $-\frac{3}{2}$

Q.22 Let Z_1 and Z_2 be i.i.d. $N(0, 1)$ random variables. If $Y = Z_1^2 + Z_2^2$, then $P(Y > 4)$ equals

(A) e^{-2} (B) $1 - e^{-2}$ (C) $\frac{1}{2}e^{-2}$ (D) e^{-4}

Q.23 Consider a sequence of independent Bernoulli trials with probability of success in each trial being $\frac{1}{3}$. Let X denote the number of trials required to get the second success. Then $P(X \geq 5)$ equals

(A) $\frac{3}{7}$ (B) $\frac{16}{27}$ (C) $\frac{16}{21}$ (D) $\frac{9}{13}$

Q.24 Let the joint probability density function of (X, Y) be

$$f(x, y) = \begin{cases} 2e^{-(x+y)}, & 0 < x < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

Then $P\left(X < \frac{Y}{2}\right)$ equals

(A) $\frac{1}{6}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{1}{2}$

Q.25 Let X_1, X_2, X_3, X_4, X_5 be a random sample from $N(0, 1)$ distribution and let $W = \frac{X_1^2}{X_2^2 + X_3^2 + X_4^2 + X_5^2}$. Then $E(W)$ equals

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{4}$ (D) $\frac{1}{5}$

Q.26 Let $X_1, X_2, \dots, X_n$ be a random sample from $N(\mu_1, \sigma^2)$ distribution and $Y_1, Y_2, \dots, Y_m$ be a random sample from $N(\mu_2, \sigma^2)$ distribution, where $\mu_i \in \mathbb{R}, i = 1, 2$ and $\sigma > 0$. Suppose that the two random samples are independent. Define

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad W = \frac{\sqrt{mn}\ (\bar{X} - \mu_1)}{\sqrt{\sum_{i=1}^{m}(Y_i - \mu_2)^2}}$$

Then which one of the following statements is TRUE for all positive integers m and n?

(A) $W \sim t_m$ (B) $W \sim t_n$

(C) $W^2 \sim F_{m,1}$ (D) $W^2 \sim F_{m,n}$

Q.27 Let $X_1, X_2, \dots, X_n$ be a random sample from $U(\theta - 0.5, \theta + 0.5)$ distribution, where $\theta \in \mathbb{R}$. If $X_{(1)} = \min\{X_1, X_2, \dots, X_n\}$ and $X_{(n)} = \max\{X_1, X_2, \dots, X_n\}$, then which one of the following estimators is **NOT** a maximum likelihood estimator of θ?

(A) $\frac{1}{2}\left(X_{(1)} + X_{(n)}\right)$ (B) $\frac{1}{4}\left(3\, X_{(1)} + X_{(n)} + 1\right)$

(C) $\frac{1}{4}\left(X_{(1)} + 3X_{(n)} - 1\right)$ (D) $\frac{1}{2}\left(3\, X_{(n)} - X_{(1)} - 2\right)$

Q.28 Let $X_1, X_2, \dots, X_n$ be a random sample from $\text{Exp}(\theta)$ distribution, where $\theta \in (0, \infty)$. If $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$, then a 95% confidence interval for θ is

(A) $\left(0, \frac{\chi^2_{2n,0.95}}{n\bar{X}}\right]$ (B) $\left[\frac{\chi^2_{2n,0.95}}{n\bar{X}}, \infty\right)$

(C) $\left(0, \frac{\chi^2_{2n,0.05}}{2n\bar{X}}\right]$ (D) $\left[\frac{\chi^2_{2n,0.05}}{2n\bar{X}}, \infty\right)$

Q.29 Let $X_1, X_2, \dots, X_n$ be a random sample from $U(1, 2)$ distribution and let $Y_1, Y_2, \dots, Y_n$ be a random sample from $U(0, 1)$ distribution. Suppose that the two random samples are independent. Define

$$Z_i = \begin{cases} 1, & \text{if } X_i Y_i < 1 \\ 0, & \text{otherwise} \end{cases}, \quad i = 1, 2, \dots n$$

If $\lim\limits_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{i=1}^{n} Z_i - \theta\right| < \epsilon\right) = 1$, for all $\epsilon > 0$, then θ equals

(A) $\frac{1}{4}$ (B) $\frac{1}{2}$ (C) $\log_e \frac{3}{2}$ (D) $\log_e 2$

Q.30 Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with probability density function

$$f_\theta(x) = \begin{cases} \theta(1-x)^{\theta-1}, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \theta > 0$$

To test $H_0: \theta = 1$ against $H_1: \theta > 1$, the uniformly most powerful test of size α $(0 < \alpha < 1)$ would reject H_0 if

(A) $-\sum_{i=1}^{n} \log_e(1-X_i)^2 < \chi^2_{2n,1-\alpha}$ (B) $-\sum_{i=1}^{n} \log_e(1-X_i)^2 < \chi^2_{n,1-\alpha}$

(C) $-\sum_{i=1}^{n} \log_e(1-X_i)^2 < \chi^2_{2n,\alpha}$ (D) $-\sum_{i=1}^{n} \log_e(1-X_i)^2 < \chi^2_{n,\alpha}$

SECTION - B

MULTIPLE SELECT QUESTIONS (MSQ)

Q. 31 – Q. 40 carry two marks each.

Q.31 Let the sequence $\{x_n\}_{n\geq1}$ be given by $x_n = \sin\frac{n\pi}{6}$, $n = 1, 2, \ldots$. Then which of the following statements is/are TRUE?

(A) The sequence $\{x_n\}_{n\geq1}$ has a subsequence that converges to $\frac{1}{2}$

(B) $\limsup_{n\to\infty} x_n = 1$

(C) $\liminf_{n\to\infty} x_n = -1$

(D) The sequence $\{x_n\}_{n\geq1}$ has a subsequence that converges to $\frac{1}{\sqrt{2}}$

Q.32 Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = x^2(2-y) - y^3 + 3y^2 + 9y$, where $(x,y) \in \mathbb{R}^2$. Which of the following is/are saddle point(s) of f?

(A) $(0,-1)$ (B) $(0,3)$ (C) $(3,2)$ (D) $(-3,2)$

Q.33 The arc length of the parabola $y^2 = 2x$ intercepted between the points of intersection of the parabola $y^2 = 2x$ and the straight line $y = 2x$ equals

(A) $\int_0^1 \sqrt{1+y^2}\, dy$ (B) $\int_0^1 \sqrt{1+4y^2}\, dy$

(C) $\int_0^{1/2} \frac{\sqrt{1+2x}}{\sqrt{2x}}\, dx$ (D) $\int_0^{1/2} \frac{\sqrt{1+4x}}{\sqrt{2x}}\, dx$

Q.34 For real constants a and b, let

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix}$$

be an orthogonal matrix. Then which of the following statements is/are always TRUE?

(A) $a + b = 0$ (B) $b = \sqrt{1 - a^2}$ (C) $ab = -\frac{1}{2}$ (D) $M^2 = I_2$

Q.35 Consider a sequence of independent Bernoulli trials with probability of success in each trial being $\frac{1}{5}$. Then which of the following statements is/are TRUE?

(A) Expected number of trials required to get the first success is 5

(B) Expected number of successes in first 50 trials is 10

(C) Expected number of failures preceding the first success is 4

(D) Expected number of trials required to get the second success is 10

Q.36 Let (X, Y) have the joint probability mass function

$$f(x,y) = \begin{cases} \binom{x+1}{y}\binom{16}{x}\left(\frac{1}{6}\right)^{y}\left(\frac{5}{6}\right)^{x+1-y}\left(\frac{1}{2}\right)^{16}, & y = 0,1,\dots,x+1;\ x = 0,1,\dots,16 \\ 0, & \text{otherwise} \end{cases}$$

Then which of the following statements is/are TRUE?

(A) $E(Y) = \frac{3}{2}$ (B) $\mathrm{Var}(Y) = \frac{49}{36}$ (C) $E(XY) = \frac{37}{3}$ (D) $\mathrm{Var}(X) = 3$

Q.37 Let X_1, X_2, X_3 be i.i.d. $N(0,1)$ random variables. Then which of the following statements is/are TRUE?

(A) $\frac{\sqrt{2}\, X_1}{\sqrt{X_2^2 + X_3^2}} \sim t_2$

(B) $\frac{\sqrt{2}\, X_1}{|X_2 + X_3|} \sim t_1$

(C) $\frac{(X_1 - X_2)^2}{(X_1 + X_2)^2} \sim F_{1,1}$

(D) $\sum_{i=1}^{3} X_i^2 \sim \chi_2^2$

Q.38 Let $\{X_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables such that

$$P(X_1 = 0) = \frac{1}{4} = 1 - P(X_1 = 1)$$

Define

$$U_n = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad V_n = \frac{1}{n}\sum_{i=1}^{n}(1 - X_i)^2 \ , \qquad n = 1, 2, \ldots$$

Then which of the following statements is/are TRUE?

(A) $\lim_{n\to\infty} P\left(\left|U_n - \frac{3}{4}\right| < \frac{1}{100}\right) = 1$

(B) $\lim_{n\to\infty} P\left(\left|U_n - \frac{3}{4}\right| > \frac{1}{100}\right) = 0$

(C) $\lim_{n\to\infty} P\left(\sqrt{n}\ \left(U_n - \frac{3}{4}\right) \leq 1\right) = \Phi(2)$

(D) $\lim_{n\to\infty} P\left(\sqrt{n}\ \left(V_n - \frac{1}{4}\right) \leq 1\right) = \Phi\left(\frac{4}{\sqrt{3}}\right)$

Q.39 Let $X_1, X_2, \ldots, X_n$ be i.i.d. Poisson(λ) random variables, where $\lambda > 0$. Define

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

Then which of the following statements is/are TRUE?

(A) $\text{Var}\ (\bar{X}) < \text{Var}(S^2)$

(B) $\text{Var}\ (\bar{X}) = \text{Var}(S^2)$

(C) $\text{Var}\ (\bar{X})$ attains the Cramer-Rao lower bound

(D) $E(\bar{X}) = E(S^2)$

Q.40 Consider the following two probability density functions (pdfs)

$$f_0(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad f_1(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Let X be a random variable having pdf $f \in \mathcal{P} = \{f_0, f_1\}$. Consider testing $H_0: f(x) = f_0(x), \forall\ x \in [0, 1]$ against $H_1: f(x) = f_1(x), \forall\ x \in [0, 1]$ at $\alpha = 0.05$ level of significance. For which of the following observed values of random observation X, the most powerful test would reject H_0?

(A) 0.19 (B) 0.22 (C) 0.25 (D) 0.28

SECTION – C

NUMERICAL ANSWER TYPE (NAT)

Q. 41 – Q. 50 carry one mark each.

Q.41 $\lim_{n\to\infty} \frac{\left(\sqrt{n^2+1}+n\right)^2}{\sqrt[3]{n^6+1}}$ equals _______

Q.42 The maximum value of the function $y = \frac{x^2}{x^4+4}, x \in \mathbb{R}$, is ________

Q.43 The value of the integral

$$\int_0^1 \int_{y^2}^1 \frac{e^x}{\sqrt{x}}\, dx\, dy$$

equals ________ *(round off to two decimal places)*

Q.44 The rank of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$$

is ________

Q.45 Let (X, Y) have the joint probability density function

$$f(x,y) = \begin{cases} \frac{3}{4}(y-x), & 0 < x < y < 2 \\ 0, & \text{otherwise} \end{cases}$$

Then the conditional expectation $E(X|Y=1)$ equals ________ *(round off to two decimal places)*

Q.46 Let X be a random variable having the Poisson(4) distribution and let E be an event such that $P(E|X=i) = 1 - 2^{-i}, i = 0, 1, 2, \ldots$. Then $P(E)$ equals ________ *(round off to two decimal places)*

Q.47 Let X_1, X_2 and X_3 be independent random variables such that $X_1 \sim N(47, 10)$, $X_2 \sim N(55,15)$ and $X_3 \sim N(60, 14)$. Then $P(X_1 + X_2 \geq 2X_3)$ equals ________ *(round off to two decimal places)*

Q.48 Let $U \sim F_{5,8}$ and $V \sim F_{8,5}$. If $P[U > 3.69] = 0.05$, then the value of c such that $P[V > c] = 0.95$ equals ________ *(round off to two decimal places)*

Q.49 Let the sample mean based on a random sample from $\text{Exp}(\lambda)$ distribution be 3.7. Then the maximum likelihood estimate of $1 - e^{-\lambda}$ equals ________ *(round off to two decimal places)*

Q.50 Let X be a single observation drawn from $U(0, \theta)$ distribution, where $\theta \in \{1, 2\}$. To test $H_0: \theta = 1$ against $H_1: \theta = 2$ consider the test procedure that rejects H_0 if and only if $X > 0.75$. If the probabilities of Type-I and Type-II errors are α and β, respectively, then $\alpha + \beta$ equals ________ *(round off to two decimal places)*

Q. 51 – Q. 60 carry two marks each.

Q.51 Let $f: [-1, 3] \to \mathbb{R}$ be a continuous function such that f is differentiable on $(-1, 3)$, $|f'(x)| \leq \frac{3}{2}$, $\forall x \in (-1, 3)$, $f(-1) = 1$ and $f(3) = 7$. Then $f(1)$ equals ________

Q.52 Let α be the real number such that the coefficient of x^{125} in Maclaurin's series of $(x + \alpha^3)e^x$ is $\frac{28}{124!}$. Then α equals ________

Q.53 Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

Let P be a non-singular matrix such that $P^{-1}MP$ is a diagonal matrix. Then the trace of the matrix $P^{-1}M^3\,P$ equals ________

Q.54 Let P be a 3×3 matrix having characteristic roots $\lambda_1 = -\frac{2}{3}$, $\lambda_2 = 0$ and $\lambda_3 = 1$. Define $Q = 3P^3 - P^2 - P + I_3$ and $R = 3P^3 - 2P$. If $\alpha = \det(Q)$ and $\beta = \text{trace}(R)$, then $\alpha + \beta$ equals ________ *(round off to two decimal places)*

Q.55 Let X and Y be independent random variables with respective moment generating functions

$$M_X(t) = \frac{(8 + e^t)^2}{81} \quad \text{and} \quad M_Y(t) = \frac{(1 + 3e^t)^3}{64}, -\infty < t < \infty$$

Then $P(X + Y = 1)$ equals ________ *(round off to two decimal places)*

Q.56 Let X be a random variable having $U(0,10)$ distribution and $Y = X - [X]$, where $[X]$ denotes the greatest integer less than or equal to X. Then $P(Y > 0.25)$ equals ________

Q.57 A computer lab has two printers handling certain types of printing jobs. Printer-I and Printer-II handle 40% and 60% of the jobs, respectively. For a typical printing job, printing time (in minutes) of Printer-I follows $N(10, 4)$ distribution and that of Printer-II follows $U(1, 21)$ distribution. If a randomly selected printing job is found to have been completed in less than 10 minutes, then the conditional probability that it was handled by the Printer-II equals ________ *(round off to two decimal places)*

Q.58 Let $x_1 = 1, x_2 = 0, x_3 = 0, x_4 = 1, x_5 = 0, x_6 = 1$ be the data on a random sample of size 6 from $\text{Bin}(1, \theta)$ distribution, where $\theta \in (0, 1)$. Then the uniformly minimum variance unbiased estimate of $\theta(1 + \theta)$ equals ________

Q.59 Let $x_1 = 1, x_2 = 4$ be the data on a random sample of size 2 from a $\text{Poisson}(\theta)$ distribution, where $\theta \in (0, \infty)$. Let $\hat{\psi}$ be the uniformly minimum variance unbiased estimate of $\psi(\theta) = \sum_{k=4}^{\infty} \frac{e^{-\theta}\, \theta^k}{k!}$ based on the given data. Then $\hat{\psi}$ equals ________ *(round off to two decimal places)*

Q.60 Let X be a random variable having $N(\theta, 1)$ distribution, where $\theta \in \mathbb{R}$. Consider testing $H_0: \theta = 0$ against $H_1: \theta \neq 0$ at $\alpha = 0.617$ level of significance. The power of the likelihood ratio test at $\theta = 1$ equals ________ *(round off to two decimal places)*

END OF THE QUESTION PAPER